Buckling of Orthotropic Circular Plates

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Exact closed form solutions in terms of Bessel and Lommel functions are obtained for the symmetric buckling of cylindrically aeolotropic circular plates. Closed form solutions are also obtained for the first asymmetric mode of buckling. Further, a series solution is proposed for higher asymmetric modes of buckling. On the basis of the closed form solutions, simple characteristic expressions to determine buckling loads are developed for fixed and supported edge conditions. The numerical results are compared with those previously obtained from series solutions.

Nomenclature

a	= radius of plate
a_1	$= S_{12}/S_{11}$
a_3	$= S_{66}/S_{11}$
D_r	$= S_{11}h^3/12$, flexural rigidity of the plate in the radial direction
h	= thickness of the plate
$J_q(z)$	= Bessel function of first kind of order q and argument z
k	$= (S_{22}/S_{11})^{1/2}$, orthotropic constant
m	= number of nodal diameters
$N_r, N_\Theta, N_{r\Theta}$	= normal and tangential stress resultants
r, Θ	= polar coordinates
s	= number of nodal circles
$S\mu_1, \mu_2(z)$	= Lommel's function defined by Eq. (69) on p. 40 of Ref. 6
S_{ij}	= orthotropic elastic stiffnesses defined in Eq. (1)
$\bar{w}$	$= w/a$, nondimensionalized deflection
ν	= Poisson's ratio of an isotropic material
σ_o	= applied normal stress at the outer edge
ρ	$= r/a$, nondimensionalized radius

Introduction

In the past, most of the structural components were made out of ferrous and nonferrous metals and their alloys which were considered as isotropic in behavior. But due to the rapid advance in the field of material science during the last two decades, structural components made of materials exhibiting various types of anisotropy are very effectively used to meet new design specifications. One such case is that of circular plates with cylindrical orthotropy, which are frequently used in the design of aerospace structural components. Stability aspect of such a plate is one of the important design factors.

The problem of elastic stability of orthotropic circular plates was first discussed by Woinowsky-Krieger.[1,2] Subsequently Mossakowski[3] gave a more detailed analysis by using an infinite series to represent the buckled mode shapes. On the basis of his approximate solution he has also suggested some empirical formulae. Apparently, not knowing the work of Mossakowski, Pandalai and Patel[4] also developed series solutions for the symmetric buckling. In the present paper, exact closed form solutions in terms of Bessel and Lommel functions are obtained for the symmetric and first asymmetric modes of buckling. Characteristic expressions for these modes of buckling are shown to be very simple expressions. A series solution is proposed for higher asymmetric modes. Numerical results are given for clamped and supported plates.

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Governing Buckling Equation

Assuming a stress-strain relation in the form

$$\begin{aligned} \sigma r &= S_{11}\varepsilon_r + S_{12}\varepsilon_\Theta \\ \sigma_\Theta &= S_{12}\varepsilon_r + S_{22}\varepsilon_\Theta \\ \tau_{r\Theta} &= S_{66}\gamma_{r\Theta} \end{aligned} \tag{1}$$

and using the classical plate theory, the governing buckling equation of a cylindrically aeolotropic circular plate can be written as

$$\frac{h^3}{12}\left[S_{11}\frac{\partial^4 w}{\partial r^4} + \frac{(2S_{12}+4S_{66})}{r^2}\frac{\partial^4 w}{\partial r^2\partial\Theta^2} + \frac{S_{22}}{r^4}\frac{\partial^4 w}{\partial\Theta^4} + \frac{2S_{11}}{r^3}\frac{\partial^3 w}{\partial r^3} - \left(\frac{2S_{12}+4S_{66}}{r^3}\right)\frac{\partial^3 w}{\partial r\,\partial\Theta^2} - \frac{S_{22}}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{(2S_{12}+4S_{66}+2S_{22})}{r^4}\frac{\partial^2 w}{\partial\Theta^2} + \frac{S_{22}}{r^3}\frac{\partial w}{\partial r}\right] = -N_r\frac{\partial^2 w}{\partial r^2} - N_\Theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\Theta^2}\right) - 2N_{r\Theta}\left(\frac{1}{r^2}\frac{\partial w}{\partial\Theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\Theta}\right)\ldots \tag{2}$$

For axisymmetric radial in-plane prebuckling loading, the plane-stress elasticity solution will give

$$N_r = h\sigma_o(r/a)^{k-1}, \quad N_\Theta = kh\sigma_o(r/a)^{k-1}, \quad N_{r\Theta} = 0 \tag{3}$$

For future convenience and generality Eq. (2) can be put in nondimensional form as

$$\frac{\partial^4\bar{w}}{\partial\rho^4} + \left(\frac{2a_1+4a_3}{\rho^2}\right)\frac{\partial^4\bar{w}}{\partial\rho^2\partial\Theta^2} + \frac{k^2}{\rho^4}\frac{\partial^4\bar{w}}{\partial\Theta^4} + \frac{2}{\rho}\frac{\partial^3\bar{w}}{\partial\rho^3} - \left(\frac{2a_1+4a_3}{\rho^3}\right)\frac{\partial^3\bar{w}}{\partial\rho\,\partial\Theta^2} - \frac{k^2}{\rho^2}\frac{\partial^2\bar{w}}{\partial\rho^2} + \left(\frac{2a_1+2k^2+4a_3}{\rho^4}\right)\frac{\partial^2\bar{w}}{\partial\Theta^2} + \frac{k^2}{\rho^3}\frac{\partial\bar{w}}{\partial\rho} + \frac{12a^2}{S_{11}h^3}\left[N_\rho\frac{\partial^2\bar{w}}{\partial\rho^2} + N_\Theta\left(\frac{1}{\rho}\frac{\partial\bar{w}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\bar{w}}{\partial\Theta^2}\right) + 2N_{\rho\Theta}\left(\frac{1}{\rho^2}\frac{\partial\bar{w}}{\partial\Theta} - \frac{1}{\rho}\frac{\partial^2\bar{w}}{\partial\rho\,\partial\Theta}\right)\right] = 0 \tag{4}$$

Now, assuming a deflection function such as

$$\bar{w} = W(\rho)\cos m\Theta \tag{5}$$

Eq. (4) can be reduced to an ordinary differential equation with variable coefficients

$$\frac{d^4W}{d\rho^4} + \frac{2}{\rho}\frac{d^3W}{d\rho^3} - \frac{\lambda^2}{\rho^2}\frac{d^2W}{d\rho^2} + \frac{\lambda^2}{\rho^3}\frac{dW}{d\rho} + \frac{\varepsilon^2}{\rho^4}W + \mu^{k+1}\left[\rho^{k-1}\frac{d^2W}{d\rho^2} + k\rho^{k-2}\frac{dW}{d\rho} - km^2\rho^{k-3}W\right] = 0 \tag{6}$$

with

$$\begin{aligned} \lambda^2 &= m^2(2a_1+4a_3)+k^2 \\ \varepsilon^2 &= m^4k^2 - m^2(2a_1+2k^2+4a_3) \\ \mu^{k+1} &= h\sigma_o a^2/D_r \end{aligned} \tag{7}$$

Symmetric Buckling

In the case of symmetric buckling ($m = 0$), Eq. (6) reduces to

$$\frac{d^4W}{d\rho^4}+\frac{2}{\rho}\frac{d^3W}{d\rho^3}-\frac{k^2}{\rho^2}\frac{d^2W}{d\rho^2}+\frac{k^2}{\rho^3}\frac{dW}{d\rho}+\mu^{k+1}\left[\rho^{k-1}\frac{d^2W}{d\rho^2}+k\rho^{k-2}\frac{dW}{d\rho}\right]=0 \tag{8}$$

which can be factorized as

$$\left(\frac{d}{d\rho}+\frac{1}{\rho}\right)\left(\frac{d^3W}{d\rho^3}+\frac{1}{\rho}\frac{d^2W}{d\rho^2}-\frac{k^2}{\rho^2}\frac{dW}{d\rho}+\mu^{k+1}\rho^{k-1}\frac{dW}{d\rho}\right)=0 \tag{9}$$

Now, the solution to Eq. (9) is found to be[5,6]

$$W=A\int_0^\rho S_{\phi,\Lambda}(\delta\rho^\gamma)\,d\rho+B\sum_{n=0}^{\infty}J_{(\Lambda+2n+1)}(\delta\rho^\gamma)+C\sum_{n=0}^{\infty}J_{(-\Lambda+2n+1)}(\delta\rho^\gamma)+D \tag{10}$$

where

$$\gamma=(k+1)/2,\quad \delta=2\mu^\gamma/(k+1),\quad \Lambda=2k/(k+1),\quad \phi=(1-k)/(1+k) \quad \ldots \tag{11}$$

When the plate is isotropic, viz., $k = 1$, Eq. (10) reduces to the well-known solution

$$W=A\log(\rho)+BJ_o(\mu\rho)+CY_o(\mu\rho)+D \tag{12}$$

In order to ensure finite deflection at the center, the coefficients A and C should be set equal to zero. Then the expression (10) for the nondimensionalized shape is given as

$$\bar{W}=B\sum_{n=0}^{\infty}J_{(\Lambda+2n+1)}(\delta\rho^\gamma)+D \tag{13}$$

Now, substituion of expression (13) into the boundary conditions will lead to the buckling equation for the determination of critical external forces applied at the edges. For a plate which is clamped, the conditions that should be satisfied are the displacement and radial slopes to vanish at the edge. On applying these conditions, the buckling equation for a clamped orthotropic plate is seen to be

$$J_\Lambda(\delta)=0 \tag{14}$$

Applying the conditions for a simply supported plate, that is, displacement and radial moment to vanish at the edge, gives rise to

$$(k+a_1)J_\Lambda(\delta)-\gamma\delta J_{\Lambda+1}(\delta)=0 \tag{15}$$

as the characteristic equation. On setting $k = 1$ and $a_1 = \nu$, expressions (14) and (15) will reduce to that of isotropic case.

Asymmetric Buckling

The problem of asymmetric buckling has been reduced to obtaining a satisfactory solution to Eq. (6). A closed form solution to Eq. (6) is not seen to be easily possible. However, for the first asymmetric mode, that is for $m = 1$, it is again possible to obtain a similar solution as in the case of symmetric buckling. For $m = 1$, Eq. (6) will take the following form

$$\frac{d^4W}{d\rho^4}+\frac{2}{\rho}\frac{d^3W}{d\rho^3}-\frac{\lambda_1^2}{\rho^2}\frac{d^2W}{d\rho^2}+\frac{\lambda_1^2}{\rho^3}\frac{dW}{d\rho}-\frac{\lambda_1^2}{\rho^4}W+\mu^{k+1}\left[\rho^{k-1}\frac{d^2W}{d\rho^2}+k\rho^{k-2}\frac{dW}{d\rho}-k\rho^{k-3}W\right]=0 \tag{16}$$

where

$$\lambda_1^2=2a_1+4a_3+k^2 \tag{17}$$

Introducing a transformation

$$W=\rho w_1 \tag{18}$$

Eq. (16) can be written as

$$\left(\frac{d}{d\rho}+\frac{3}{\rho}\right)\left(\frac{d^3w_1}{d\rho^3}+\frac{3}{\rho}\frac{d^2w_1}{d\rho^2}-\frac{\lambda_1^2}{\rho^2}\frac{dw_1}{d\rho}+\mu^{k+1}\rho^{k-1}\frac{dw_1}{d\rho}\right)=0 \tag{19}$$

Solution to Eq. (19) is

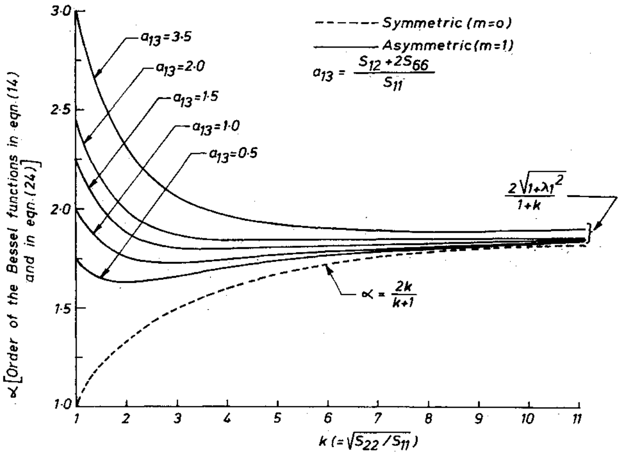

Fig. 1 Variation of the order of the Bessel function with material properties.

$$w_1=A_1\int_0^\rho \rho^{-1}S_{-1,\alpha}(\delta\rho^\gamma)\,d\rho+B_1\int_0^\rho \rho^{-1}J_\alpha(\delta\rho^\gamma)\,d\rho+C_1\int_0^\rho \rho^{-1}J_{-\alpha}(\delta\rho^\gamma)\,d\rho+D_1 \tag{20}$$

where

$$\alpha=2(1+\lambda_1^2)^{1/2}/(1+k) \tag{20a}$$

Hence

$$\bar{w}=\left[A_1\rho\int_0^\rho \rho^{-1}S_{-1,\alpha}(\delta\rho^\gamma)\,d\rho+B_1\rho\int_0^\rho \rho^{-1}J_\alpha(\delta\rho^\gamma)\,d\rho+C_1\rho\int_0^\rho \rho^{-1}J_{-\alpha}(\delta\rho^\gamma)\,d\rho+D_1\rho\right]\cos\Theta \tag{21}$$

When the material is isotropic, Eq. (21) reduces to the well-known solution

$$\bar{w}=[A_1\rho^{-1}+B_1J_1(\mu\rho)+C_1Y_1(\mu\rho)+D_1\rho]\cos\Theta \tag{22}$$

For the deflection to be finite at the center, the coefficients A_1 and C_1 should be set equal to zero. Hence, the nondimensionalized deflection function is

$$\bar{w}=\left[B_1\rho\int_0^\rho \rho^{-1}J_\alpha(\delta\rho^\gamma)\,d\rho+D_1\rho\right]\cos\Theta \tag{23}$$

Applying clamped edge conditions on Eq. (23) will give rise to the following buckling equation

$$J_\alpha(\delta)=0 \tag{24}$$

Similarly, for a simply supported plate the buckling equation is

$$\delta\gamma J_{\alpha+1}(\delta)-(\alpha+1+a_1)J_\alpha(\delta)=0 \tag{25}$$

On setting $k = 1$ and $a_1 = \nu$, Eqs. (24) and (25) will reduce to that of isotropic case.

When $m \geqq 2$, viz., the number of nodal diameters formed are more than one, a series solution is obtained. Using the transformation $\rho = e^u$, Eq. (6) can be reduced to

$$\frac{d^4W}{du^4}-4\frac{d^3W}{du^3}+(5-\lambda^2)\frac{d^2W}{du^2}+(2\lambda^2-2)\frac{dW}{du}+\varepsilon^2W+\mu^{k+1}e^{(k+1)u}\left[\frac{d^2W}{du^2}+(k-1)\frac{dW}{du}-km^2W\right]=0 \tag{26}$$

Assuming a series of the form

$$W=\sum_{j=0}^{\infty}d_j e^{[p+j(k+1)]u} \tag{27}$$

and substituting into Eq. (26), and equating all the like exponents in u to zero, we get the indicial equation

$$[p^4-4p^3+(5-\lambda^2)p^2+(2\lambda^2-2)p+\varepsilon^2]\,d_o=0 \tag{28}$$

and the recurrence relations

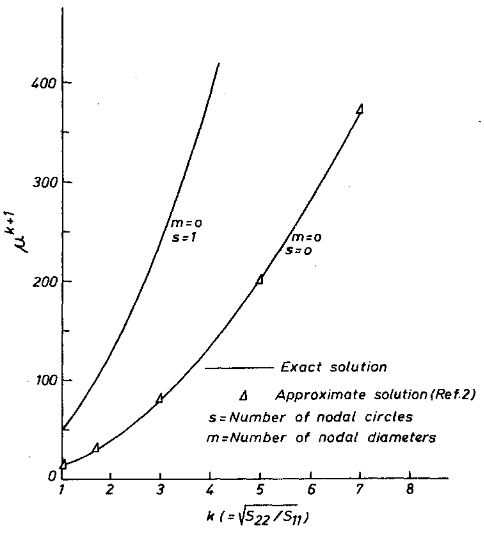

Fig. 2 Buckling parameter for the symmetric buckling of a clamped plate.

$$\{[p+j(k+1)]^4-4[p+j(k+1)]^3+(5-\lambda^2)[p+j(k+1)]^2+ (2\lambda^2-2)[p+j(k+1)]+\varepsilon^2\}\,d_j+\mu^{k+1}\{[p+(j-1)(k+1)]^2+ (k-1)[p+(j-1)(k+1)]-km^2\}\,d_{j-1}=0 \quad (29)$$

The roots of the indicial equation, (28), are

$$p_1=1+\eta+\beta, \quad p_2=1+\eta-\beta$$
$$p_3=1-\eta+\beta, \quad p_4=1-\eta-\beta \quad (30)$$

where

$$\eta=1/2[1+\lambda^2+2(\lambda^2+\varepsilon^2)^{1/2}]^{1/2} \quad (31)$$
$$\beta=1/2[1+\lambda^2-2(\lambda^2+\varepsilon^2)^{1/2}]^{1/2}$$

Using the recurrence relations (29) and the roots of Eq. (30), the coefficients d_j are determined. Hence, the solution to Eq. (6) can be written as

$$W=A_2\psi_1(\mu,\rho)+B_2\psi_2(\mu,\rho)+C_2\psi_3(\mu,\rho)+D_2\psi_4(\mu,\rho)\ldots \quad (32)$$

where

$$\psi_1(\mu,\rho)=\rho^{p_1}[{}_1d_0-{}_1d_1(\mu\rho)^{k+1}+{}_1d_2(\mu\rho)^{2(k+1)}-{}_1d_3(\mu\rho)^{3(k+1)}+\cdots]$$
$$\psi_2(\mu,\rho)=\rho^{p_2}[{}_2d_0-{}_2d_1(\mu\rho)^{k+1}+{}_2d_2(\mu\rho)^{2(k+1)}-{}_2d_3(\mu\rho)^{3(k+1)}+\cdots]$$
$$\psi_3(\mu,\rho)=\rho^{p_3}[{}_3d_0-{}_3d_1(\mu\rho)^{k+1}+{}_3d_2(\mu\rho)^{2(k+1)}-{}_3d_3(\mu\rho)^{3(k+1)}+\cdots] \quad (33)$$
$$\psi_4(\mu,\rho)=\rho^{p_4}[{}_4d_0-{}_4d_1(\mu\rho)^{k+1}+{}_4d_2(\mu\rho)^{2(k+1)}-{}_4d_3(\mu\rho)^{3(k+1)}+\cdots]$$

When the material is isotropic, Eq. (32) gives

$$W=A_2J_m(\mu\rho)+B_2\rho^m+C_2Y_{-m}(\mu\rho)+D_2\rho^{-m} \quad (34)$$

Considering the finiteness of the deflection at the center, the constants C_2 and D_2 are to be set equal to zero and the solution to Eq. (4) can be written as

$$\bar{w}=[A_2\psi_1(\mu,\rho)+B_2\psi_2(\mu,\rho)]\cos m\Theta \quad (35)$$

Applying specific boundary conditions on (35) will give the corresponding buckling equation as a 2×2 determinant. Each element of the determinant will be an infinite series. However, the series are seen to converge rapidly.

Numerical Results and Discussions

As may be seen from Eqs. (14, 15, 24 and 25), the determination of critical buckling loads essentially involves Bessel functions of fractional order. The variation of the order of the Bessel function with respect to the material properties is shown in Fig. 1. For the symmetric buckling of clamped plate, the order of the Bessel function is given by the dashed line of Fig. 1 and it shows that order α lies between 1 and 2. For the asymmetric buckling of clamped plate, the order of the Bessel function approaches 2 asymptotically as the values of k tend to become infinity.

Figure 2 shows the buckling parameter for the first two symmetric modes of clamped plates. It may be seen that for the first symmetric mode of buckling, the values calculated from approximate formulae given by Mossakowski (based on the results obtained from the infinite series solution) check very well with the exact solution. However, from Fig. 3, it may be seen that for the first asymmetric case, the values calculated from approximate solution and exact solution differ considerably. For $k=1$ and $a_{13}=3.5$, the difference between the two is 17.4%. For the supported plate, the approximate solution is in error even for the first symmetric mode (see Fig. 4). When $k=2$, the

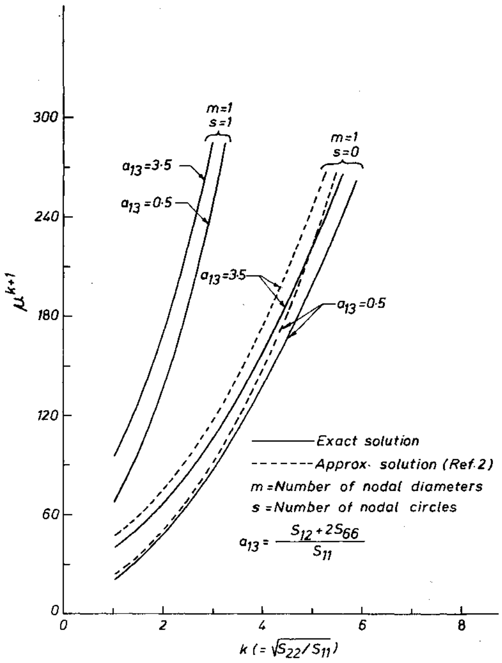

Fig. 3 Buckling parameter for the asymmetric buckling of a clamped plate.

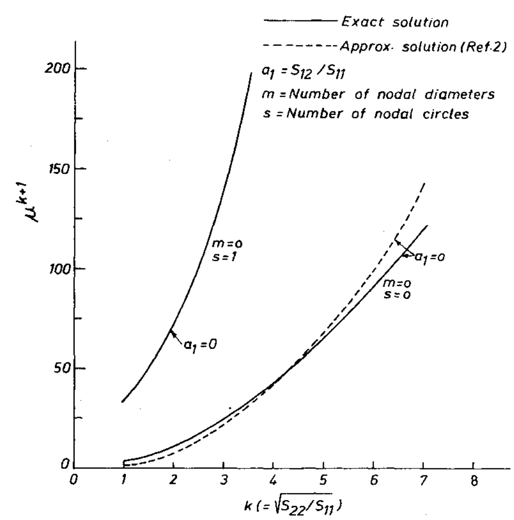

Fig. 4 Buckling parameter for the symmetric buckling of a simply supported plate.

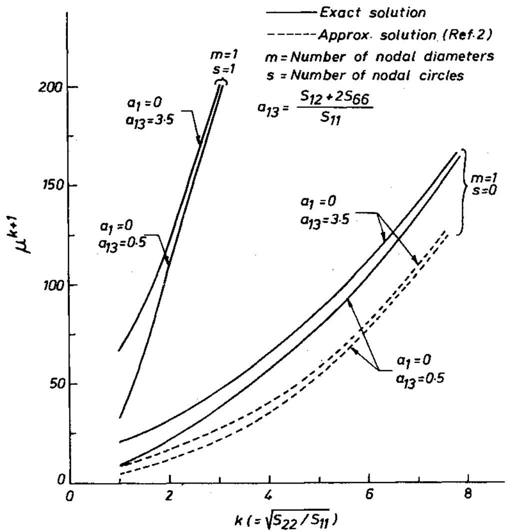

Fig. 5 Buckling parameter for the asymmetric buckling of a simply supported plate.

difference between the exact and approximate is -32% while for $k = 7$, the difference between the two is $+17.5\%$. For the first asymmetric buckling of supported plate the difference between the approximate and exact values are considerable (see Fig. 5). The difference between the two values increases with larger values of k. The maximum difference between the two is about 50.5% when $k = 1$ ($a_1 = 0$, $a_{13} = 3.5$) and the difference is 32.1% when $k = 7$.

On the basis of the present results it may be concluded that the approximate solution will be in error except for the first symmetric buckling load of clamped plates. It is also evident that the buckling equations obtained on the basis of exact solutions are so simple that approximate tables[7] and slide rule would be sufficient to calculate the buckling loads.

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